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Michaud et al.

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(54) **PORTFOLIO OPTIMIZATION BY MEANS OF META-RESAMPLED EFFICIENT FRONTIERS**

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(60) Division of application No. 11/158,267, filed on Jun. 21, 2005, now Pat. No. 7,412,414, which is a continuation-in-part of application No. 10/280,384, filed on Oct. 25, 2002, now Pat. No. 6,928,418.

(51) **Int. Cl.**
G06Q 40/00 (2006.01)

(52) **U.S. Cl.** .... **705/36 R**; 705/35; 705/37

(58) **Field of Classification Search** .... 705/36 R, 705/35, 37

See application file for complete search history.

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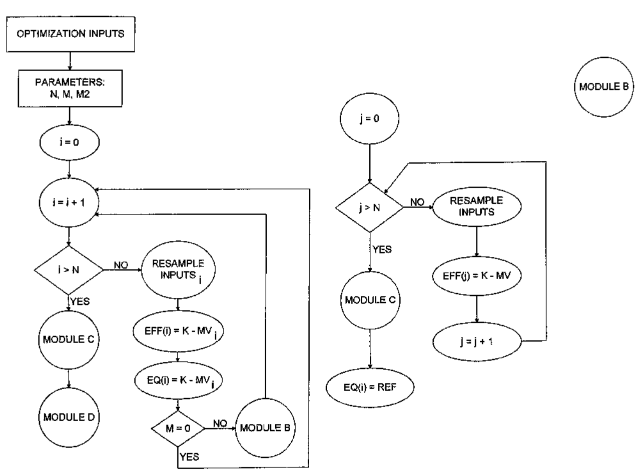
(57) **ABSTRACT**

A computer-implemented method and computer program product for selecting a portfolio weight (subject to specified constraints) for each of a plurality of assets of an optimal portfolio. A mean-variance efficient frontier is calculated based on expected return and standard deviation of return of each of the plurality of assets. Multiple sets of optimization inputs are drawn from a distribution of simulated optimization inputs consistent with the expected return and standard deviation of return of each of the assets and a specified level of forecast certainty, and then a simulated mean-variance efficient frontier is computed for each set of optimization inputs. A meta-resampled efficient frontier is determined as a statistical mean of associated portfolios, and a portfolio weight is selected for each asset according to a specified investment objective.

24 Claims, 15 Drawing Sheets



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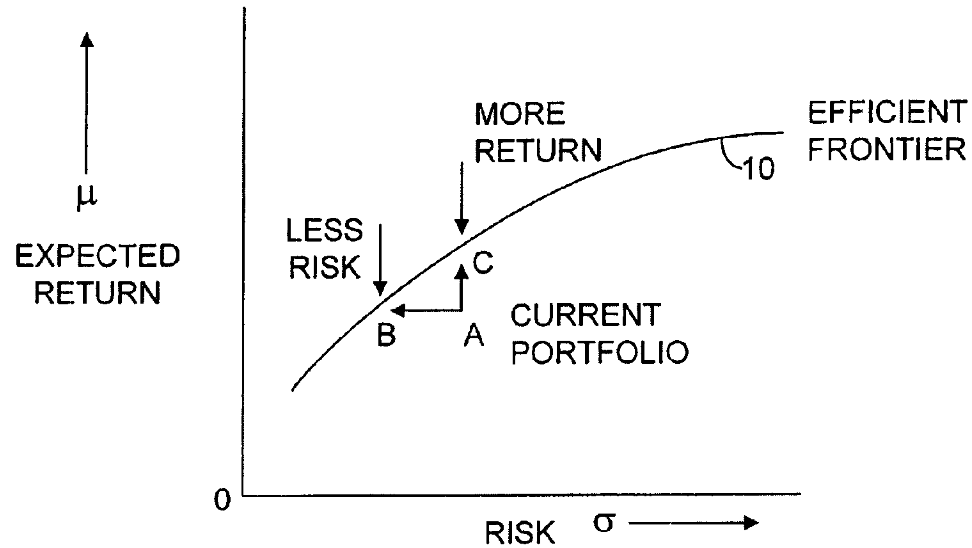


FIG. 1A
PRIOR ART

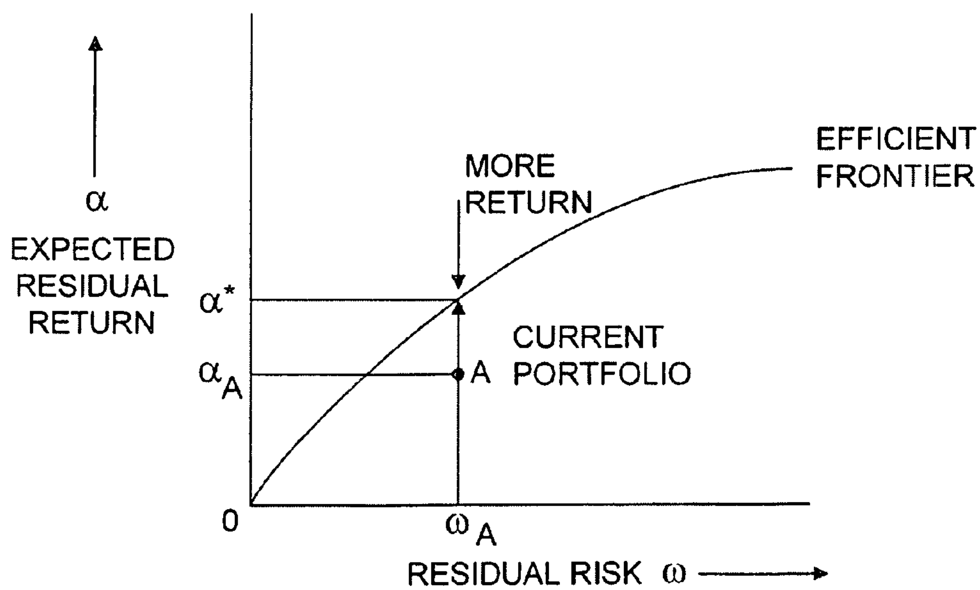


FIG. 1B
PRIOR ART

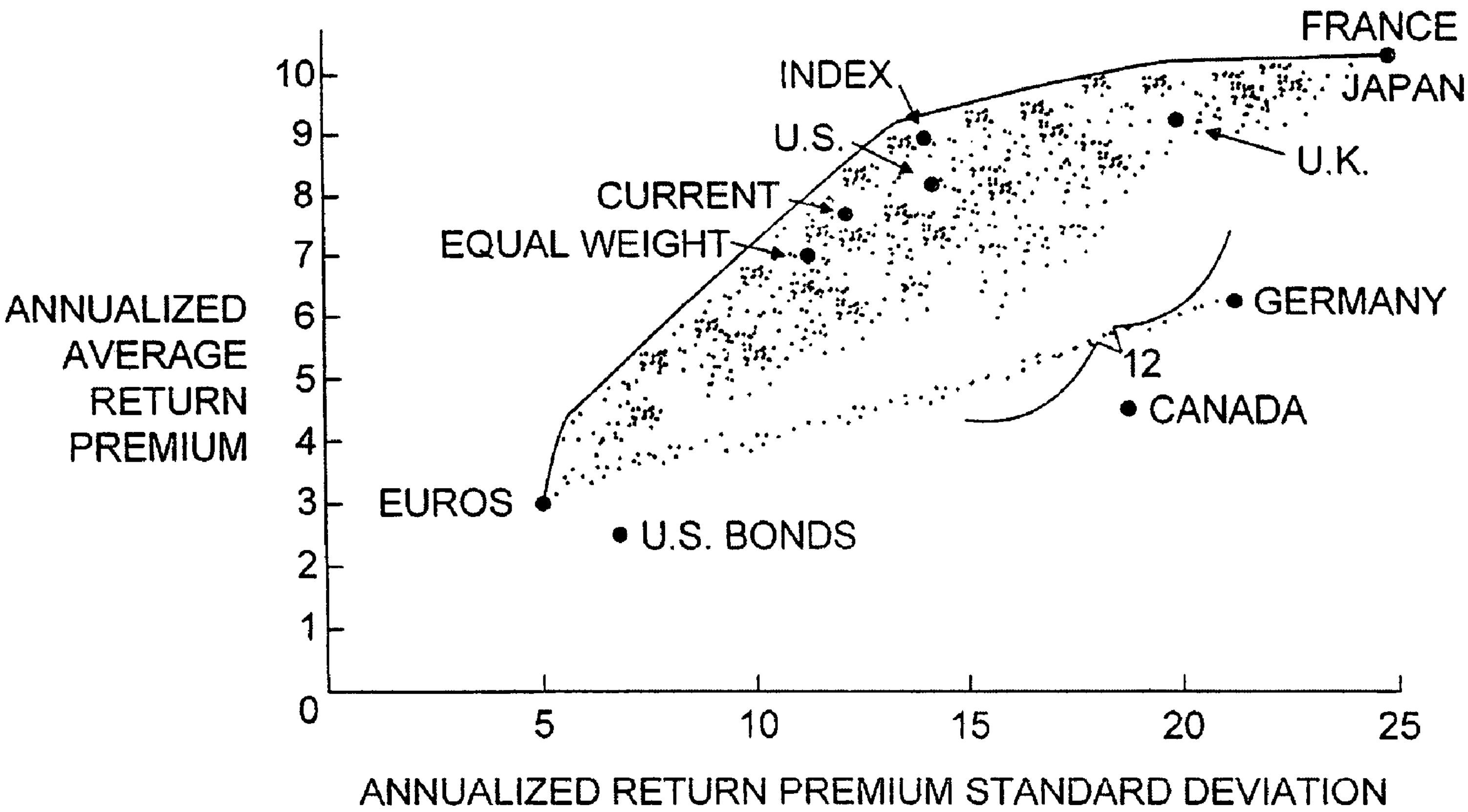


FIG. 2

PRIOR ART

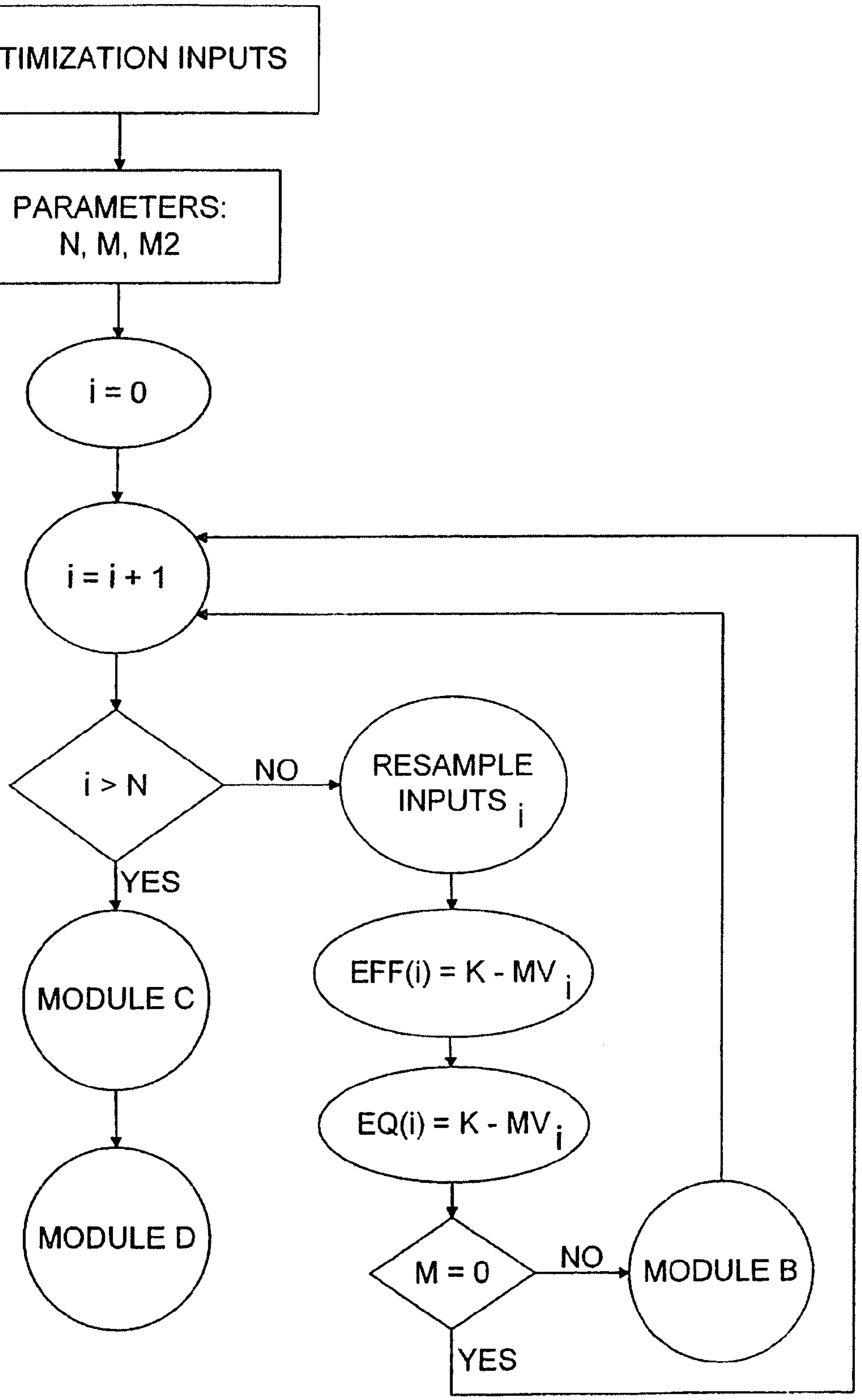

OPTIMIZATION INPUTS
PARAMETERS: N, M, M2
i = 0
i = i + 1
i > N
NO
YES
RESAMPLE INPUTS i
EFF(i) = K - MV i
EQ(i) = K - MV i
M = 0
NO
YES
MODULE B
MODULE C
MODULE D

FIG. 3A

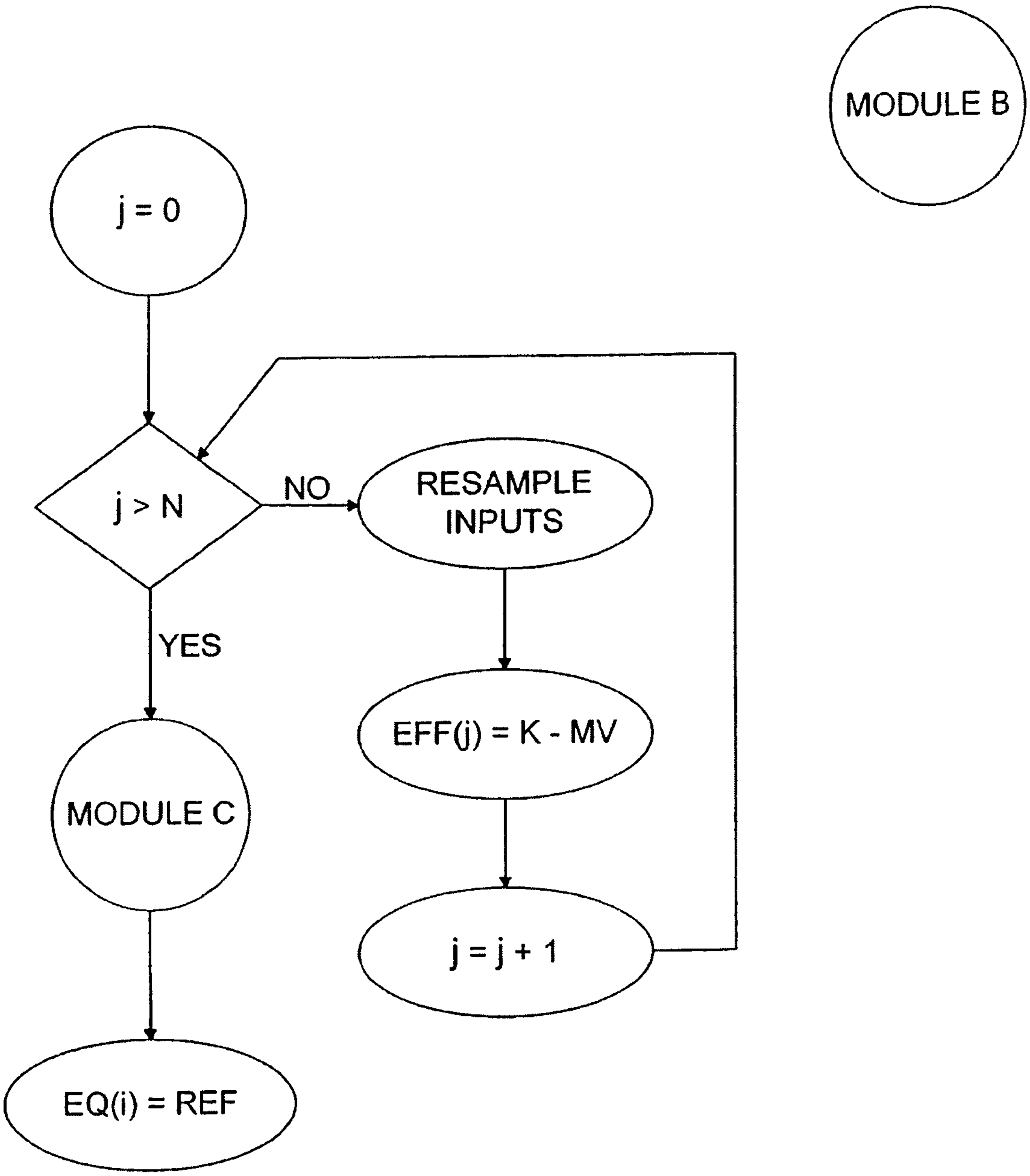

MODULE B
j = 0
j > N
NO
RESAMPLE INPUTS
YES
EFF(j) = K - MV
MODULE C
j = j + 1
EQ(i) = REF

FIG. 3B

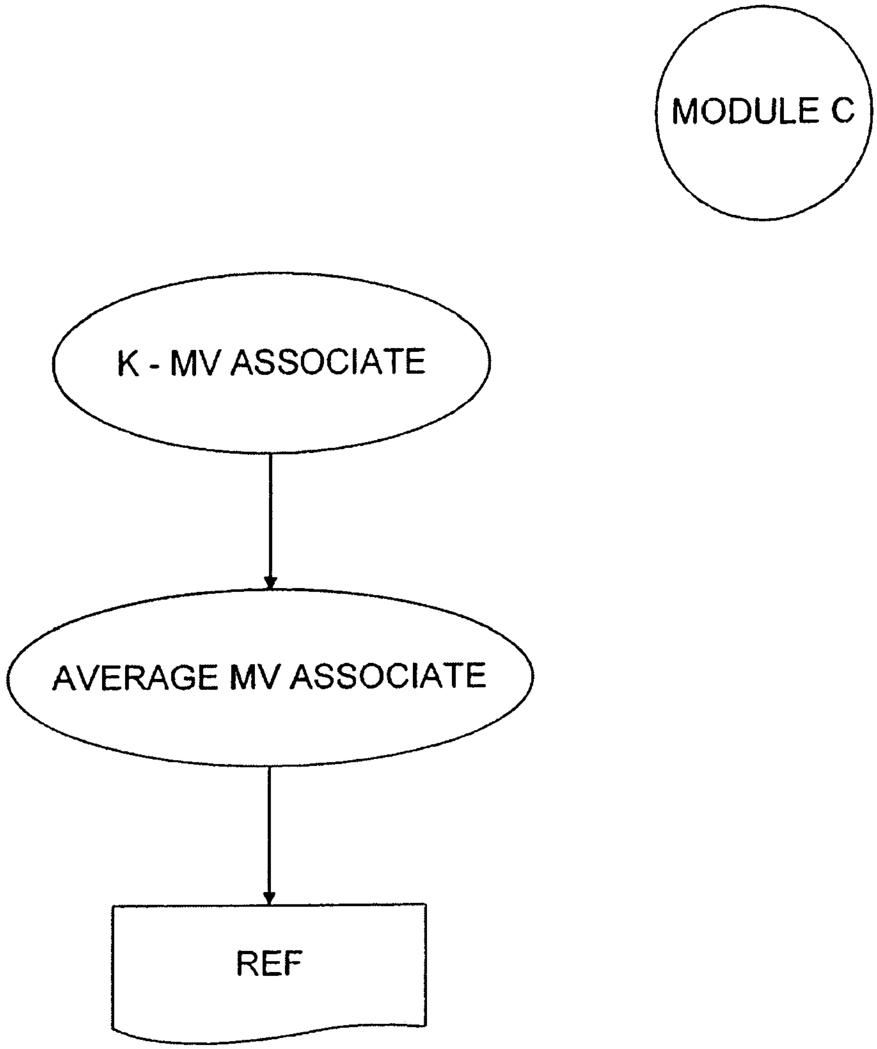


FIG. 3C

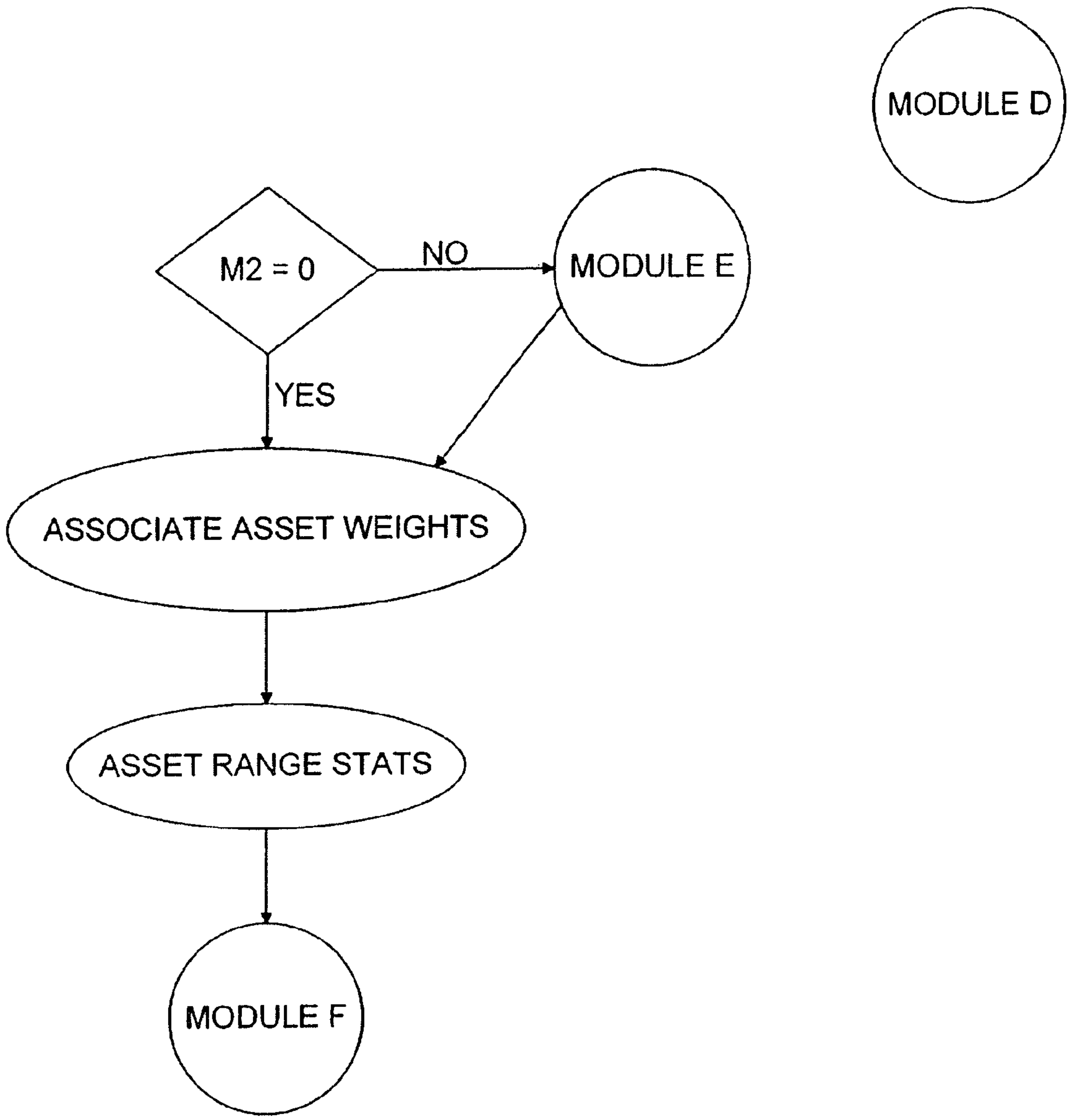

MODULE D
M2 = 0
NO
MODULE E
YES
ASSOCIATE ASSET WEIGHTS
ASSET RANGE STATS
MODULE F

FIG. 3D

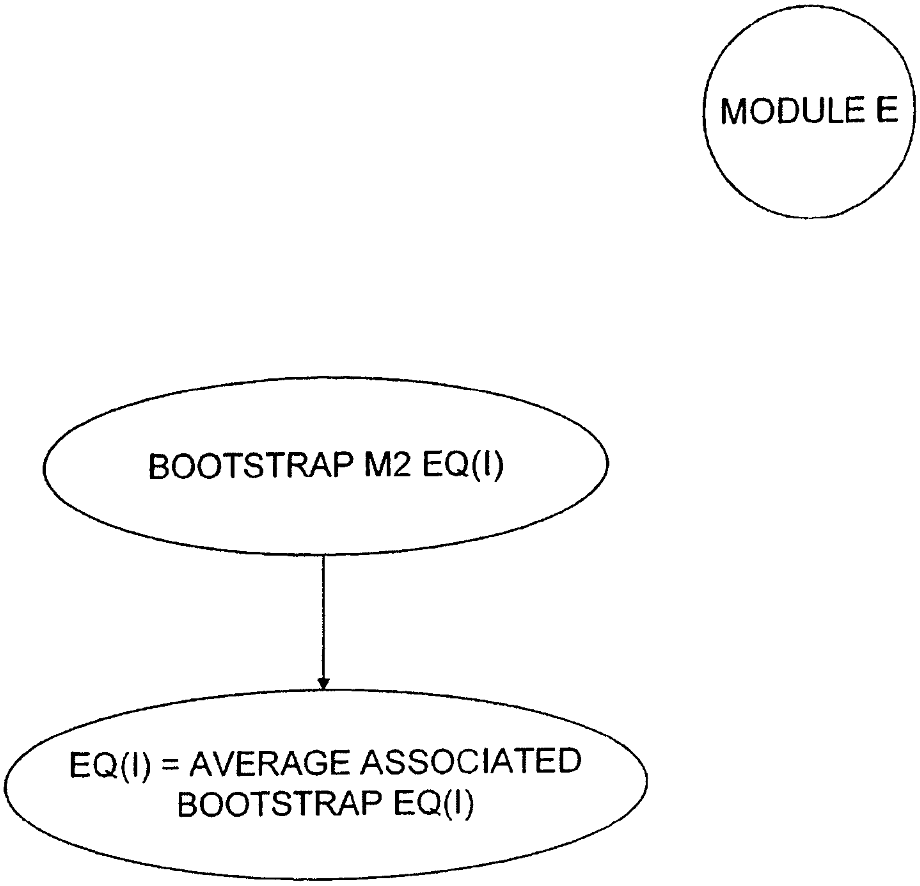

MODULE E
BOOTSTRAP M2 EQ(I)
EQ(I) = AVERAGE ASSOCIATED BOOTSTRAP EQ(I)

FIG. 3E

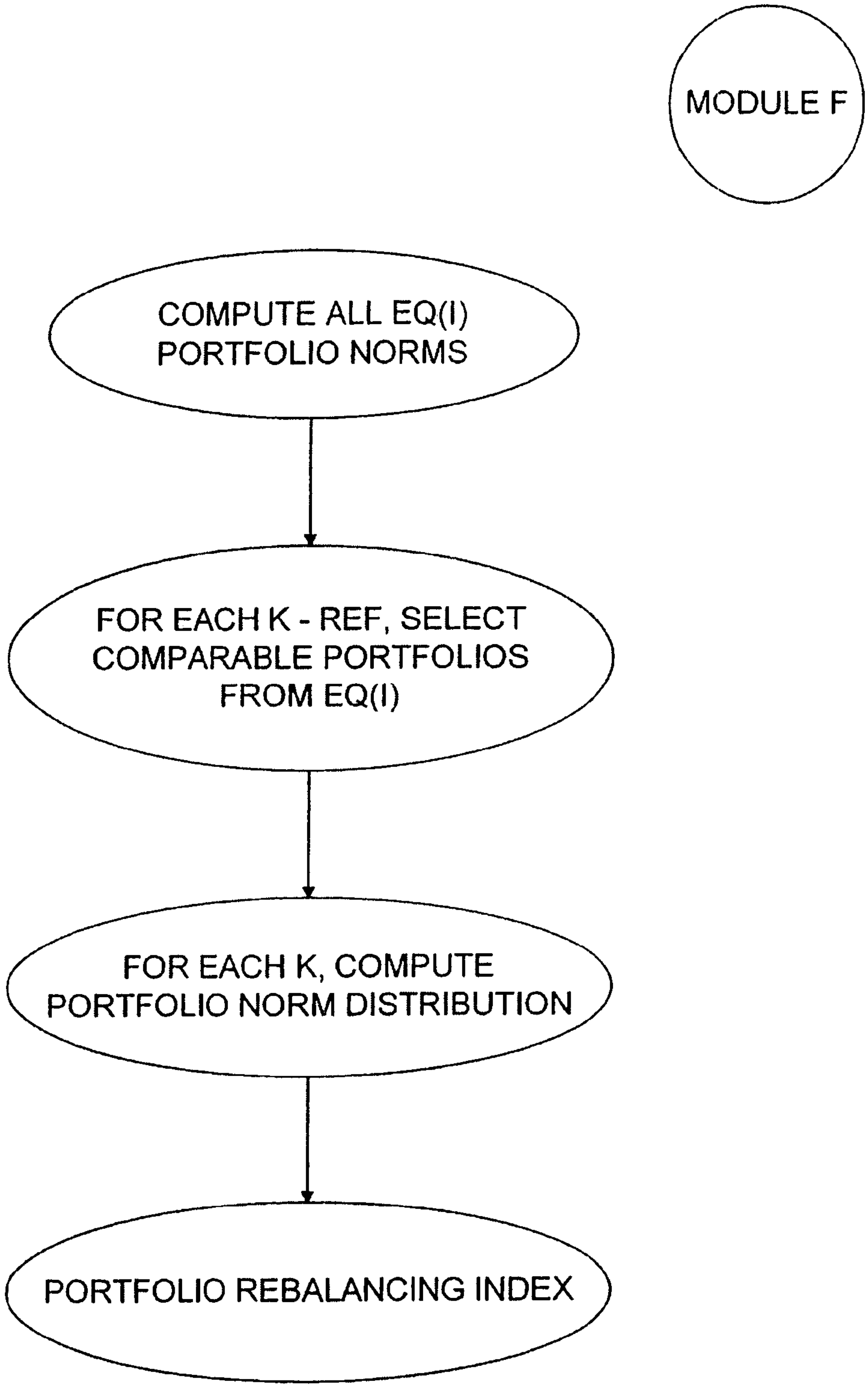

MODULE F
COMPUTE ALL EQ(I) PORTFOLIO NORMS
FOR EACH K - REF, SELECT COMPARABLE PORTFOLIOS FROM EQ(I)
FOR EACH K, COMPUTE PORTFOLIO NORM DISTRIBUTION
PORTFOLIO REBALANCING INDEX

FIG. 3F

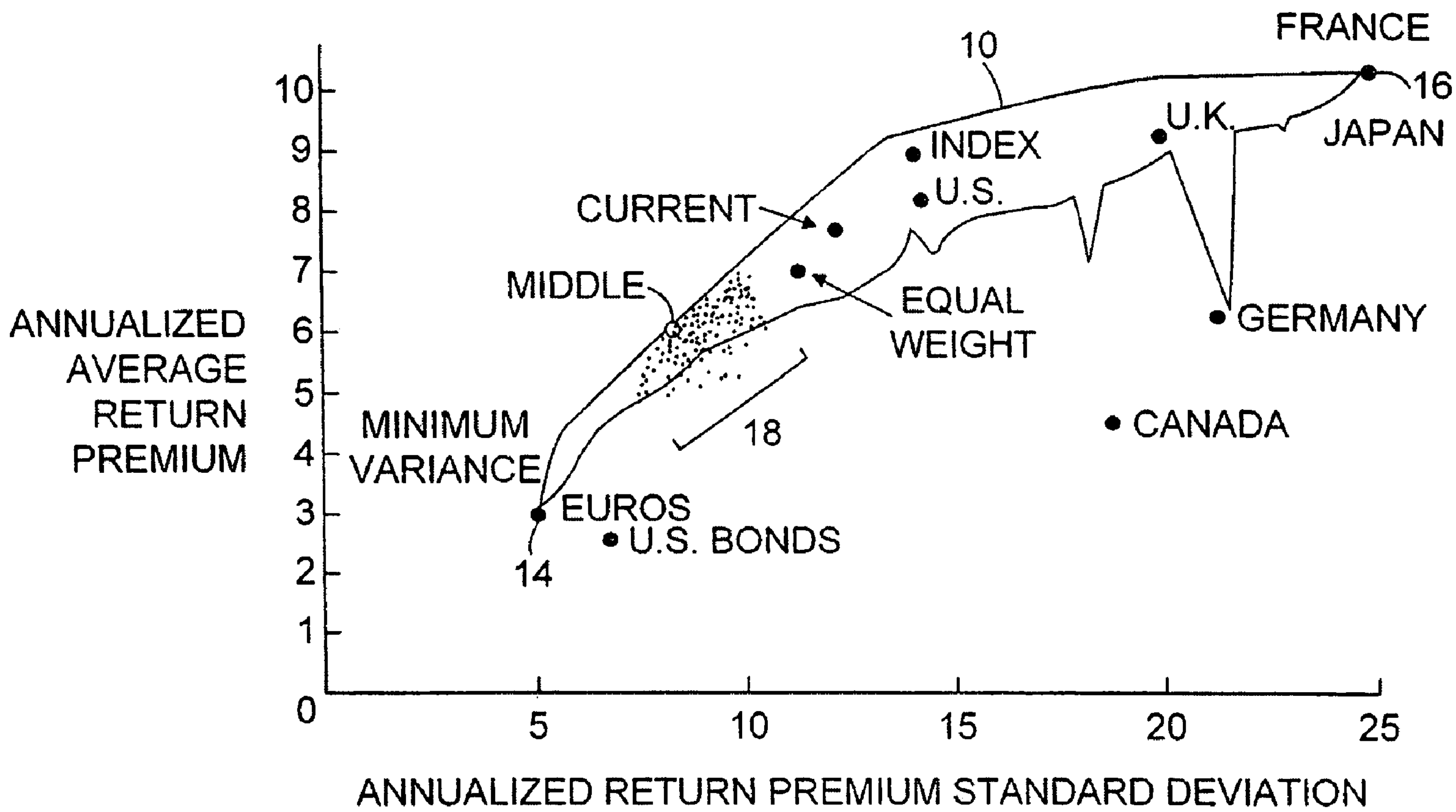

FRANCE
10
16
10
U.K.
JAPAN
9
INDEX
8
U.S.
CURRENT
7
MIDDLE
EQUAL
WEIGHT
GERMANY
ANNUALIZED
AVERAGE
RETURN
PREMIUM
6
5
18
MINIMUM
VARIANCE
4
CANADA
3
EUROS
2
U.S. BONDS
14
1
0
5
10
15
20
25
ANNUALIZED RETURN PREMIUM STANDARD DEVIATION

FIG. 4

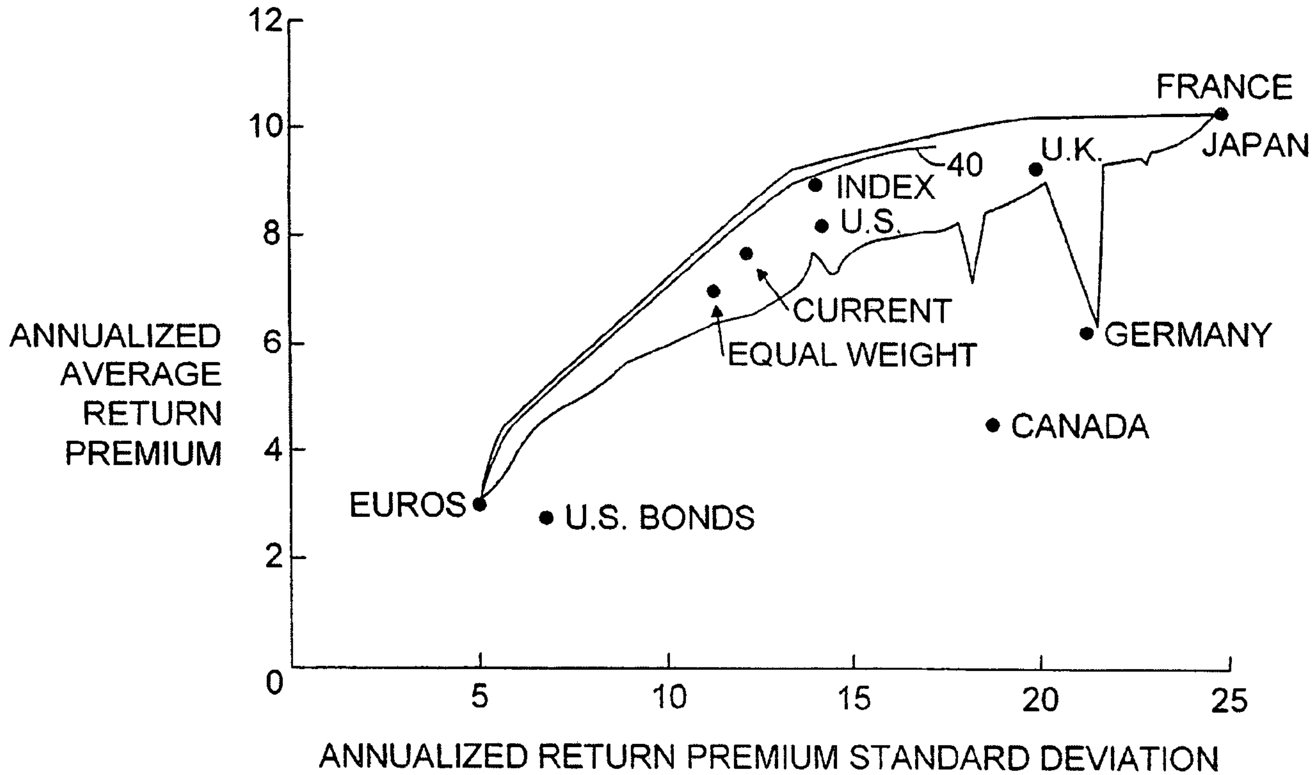
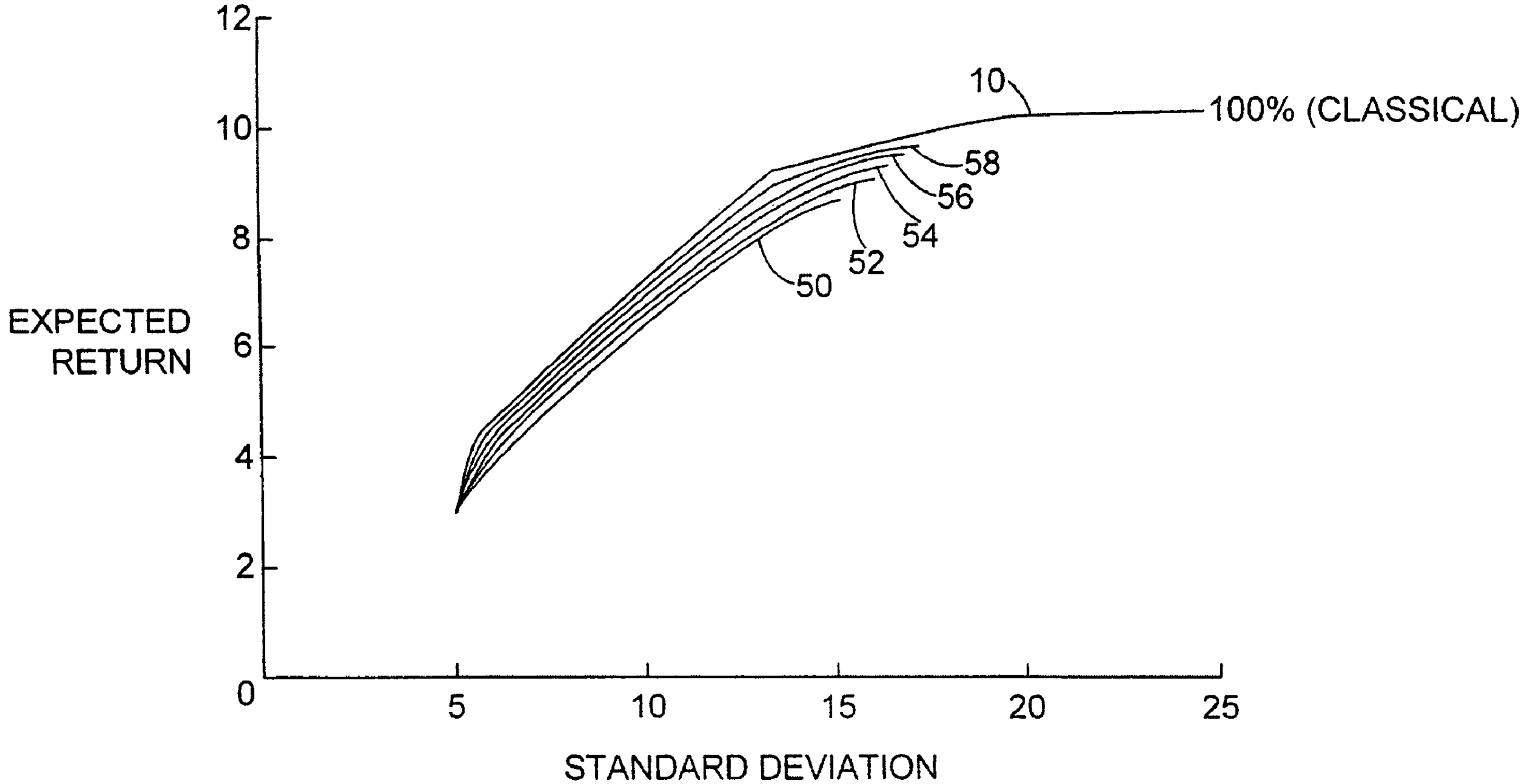

12
10
8
6
4
2
0
5
10
15
20
25
ANNUALIZED AVERAGE RETURN PREMIUM
ANNUALIZED RETURN PREMIUM STANDARD DEVIATION
FRANCE
JAPAN
U.K.
40
INDEX
U.S.
CURRENT
EQUAL WEIGHT
GERMANY
CANADA
EUROS
U.S. BONDS

FIG. 5



EXPECTED RETURN
STANDARD DEVIATION
100% (CLASSICAL)
10
58
56
54
52
50
0
2
4
6
8
10
12
5
10
15
20
25

FIG. 6

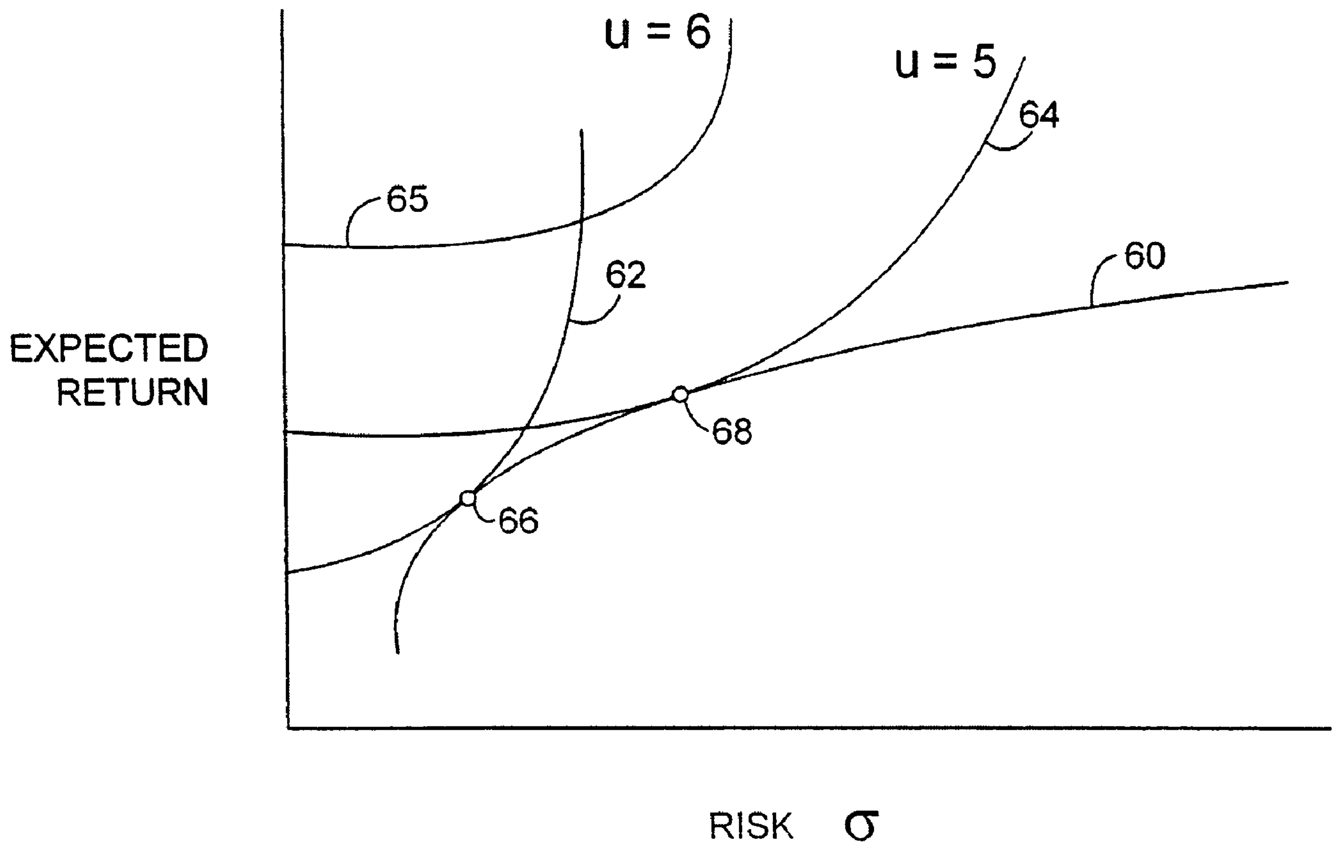


FIG. 7

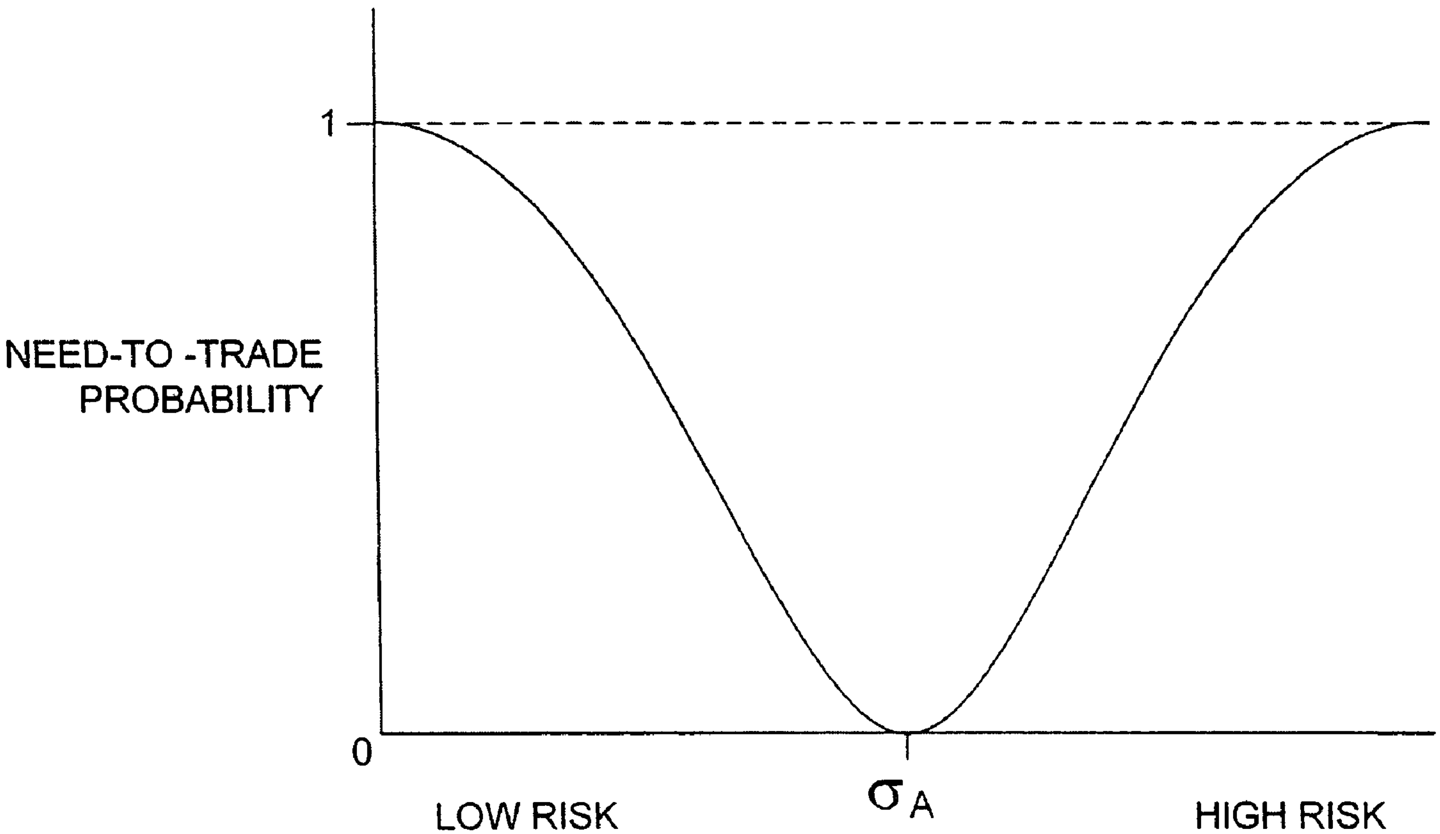
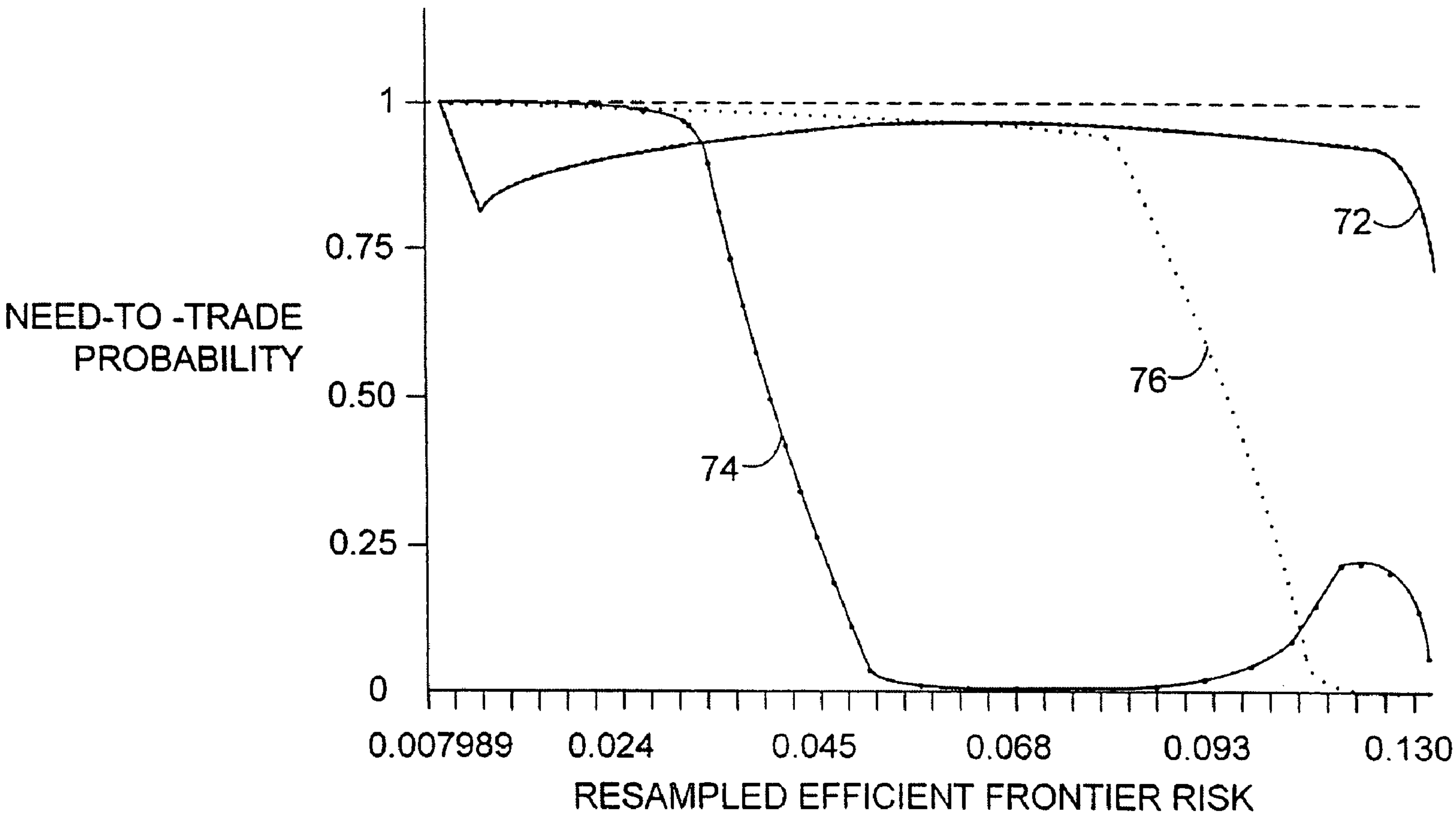


FIG. 8A



NEED-TO -TRADE PROBABILITY
1
0.75
0.50
0.25
0
0.007989
0.024
0.045
0.068
0.093
0.130
RESAMPLED EFFICIENT FRONTIER RISK
72
74
76

FIG. 8B

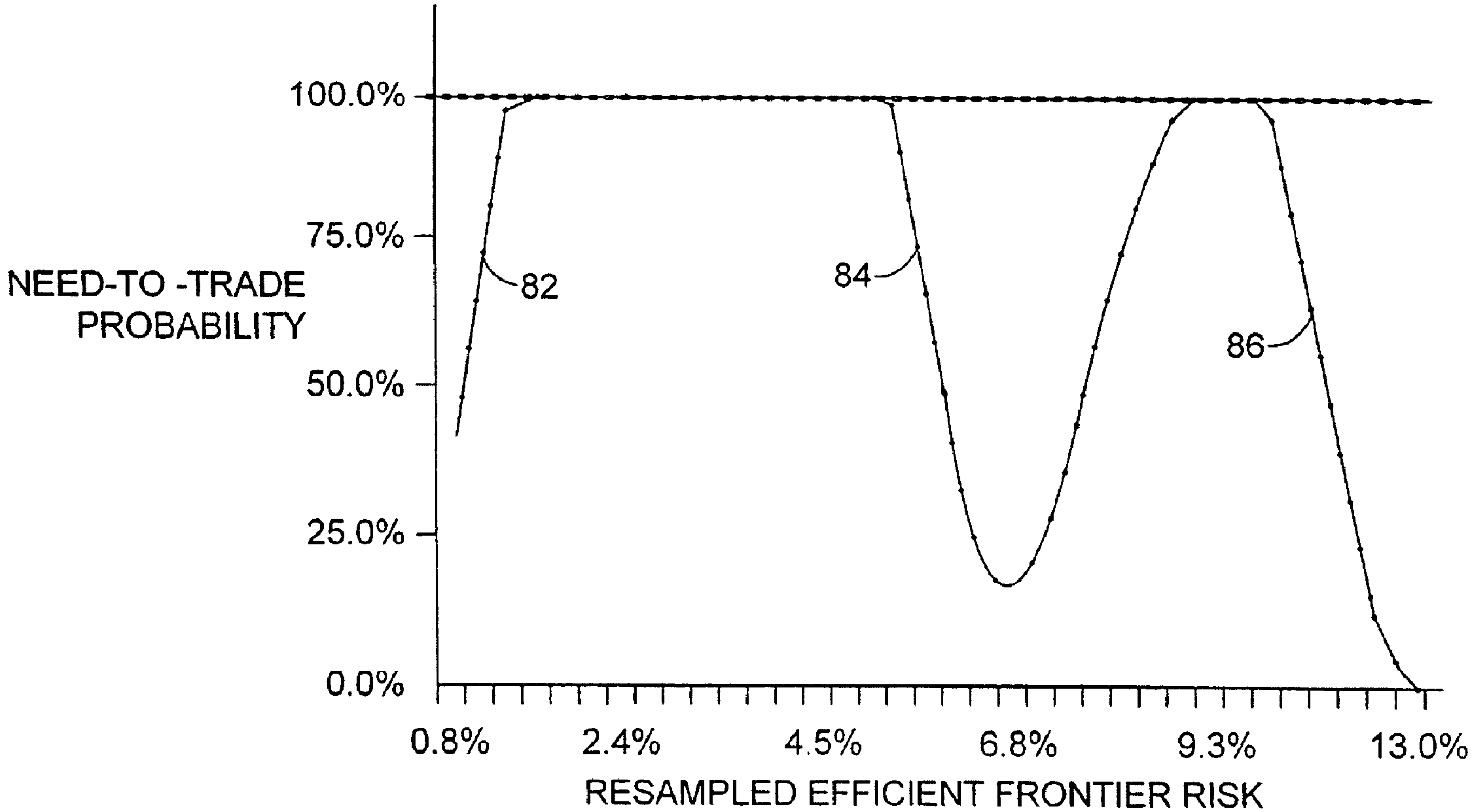

NEED-TO -TRADE PROBABILITY
100.0%
75.0%
50.0%
25.0%
0.0%
82
84
86
0.8%
2.4%
4.5%
6.8%
9.3%
13.0%
RESAMPLED EFFICIENT FRONTIER RISK

FIG. 8C

PORTFOLIO OPTIMIZATION BY MEANS OF META-RESAMPLED EFFICIENT FRONTIERS

The present application is a divisional application of U.S. patent application Ser. No. 11/158,267, filed Jun. 21, 2005 issued as U.S. Pat. No. 7,412,414, which was a continuation-in-part of U.S. Ser. No. 10/280,384, filed Oct. 25, 2002 and subsequently issued as U.S. Pat. No. 6,928,418. This application claims priority from the latter application. Both predecessor applications are incorporated herein by reference.

TECHNICAL FIELD

The present invention relates to methods for controlling the discriminatory power of statistical tests of congruence between a current portfolio of tangible or intangible assets and a target portfolio and of defining normal ranges of allocation to asset classes within a portfolio.

BACKGROUND ART

Managers of assets, such as portfolios of stocks, projects in a firm, or other assets, typically seek to maximize the expected or average return on an overall investment of funds for a given level of risk as defined in terms of variance of return, either historically or as adjusted using techniques known to persons skilled in portfolio management. Alternatively, investment goals may be directed toward residual return with respect to a benchmark as a function of residual return variance. Consequently, the terms "return" and "variance," as used in this description and in any appended claims, may encompass, equally, the residual components as understood in the art. The capital asset pricing model of Sharpe and Lintner and the arbitrage pricing theory of Ross are examples of asset evaluation theories used in computing residual returns in the field of equity pricing. Alternatively, the goal of a portfolio management strategy may be cast as the minimization of risk for a given level of expected return.

The risk assigned to a portfolio is typically expressed in terms of its variance σ_p^2 stated in terms of the weighted variances of the individual assets, as:

$$\sigma_p^2 = \sum_i \sum_j wiwj\sigma ij,$$

where w_i is the relative weight of the i-th asset within the portfolio, $\sigma_{ij}=\sigma_i\sigma_j\rho_{ij}$ is the covariance of the i-th and j-th assets, ρ_{ij} is their correlation, and σ_i is the standard deviation of the i-th asset. The portfolio standard deviation is the square root of the variance of the portfolio.

Following the classical paradigm due to Markowitz, a portfolio may be optimized, with the goal of deriving the peak average return for a given level of risk and any specified set of constraints, in order to derive a so-called "mean-variance (MV) efficient" portfolio using known techniques of linear or quadratic programming as appropriate. Techniques for incorporating multiperiod investment horizons are also known in the art. As shown in FIG. **1**A, the expected return μ for a portfolio may be plotted versus the portfolio standard deviation σ, with the locus of MV efficient portfolios as a function of portfolio standard deviation referred to as the "MV efficient frontier," and designated by the numeral **10**. Mathematical algorithms for deriving the MV efficient frontier are known in the art.

Referring to FIG. **1**B, a variation of classical Markowitz MV efficiency often used is benchmark optimization. In this case, the expected residual return α relative to a specified benchmark is considered as a function of residual return variance ω, defined as was the portfolio standard deviation σ but with respect to a residual risk. An investor with portfolio A desires to optimize expected residual return at the same level ω_A of residual risk. As before, an efficient frontier **10** is defined as the locus of all portfolios having a maximum expected residual return α of each of all possible levels of portfolio residual risk.

Known deficiencies of MV optimization as a practical tool for investment management include the instability and ambiguity of solutions. It is known that MV optimization may give rise to solutions which are both unstable with respect to small changes (within the uncertainties of the input parameters) and often non-intuitive and thus of little investment sense or value for investment purposes and with poor out-of-sample average performance. These deficiencies are known to arise due to the propensity of MV optimization as "estimation-error maximizers," as discussed in R. Michaud, "The Markowitz Optimization Enigma: Is Optimized Optimal?" *Financial Analysts Journal* (1989), which is herein incorporated by reference. In particular, MV optimization tends to overweight those assets having large statistical estimation errors associated with large estimated returns, small variances, and negative correlations, often resulting in poor ex-post performance.

Resampling of a plurality of simulations of input data statistically consistent with an expected return and expected standard deviation of return has been applied (see, for example, Broadie, "Computing efficient frontiers using estimated parameters", 45 *Annals of Operations Research* 21-58 (1993)) in efforts to overcome some of the statistical deficiencies inherent in use of sample moments alone. Comprehensive techniques based on a resampled efficient frontier are described in U.S. Pat. No. 6,003,018 (Michaud et al. '018), issued Dec. 14, 1999, and in the book, R. Michaud, *Efficient Asset Management*, (Harvard Business School Press, 1998, hereinafter "Michaud 1998"), that MV optimization is a statistical procedure, based on estimated returns subject to a statistical variance, and that, consequently, the MV efficient frontier, as defined above, is itself characterized by a variance. The Michaud patent and book are incorporated herein by reference, as are all references cited in the text of the book.

As taught in the Michaud '018 patent, an MV efficient frontier is first calculated by using standard techniques as discussed above. Since the input data are of a statistical nature (i.e., characterized by means with associated variances and other statistical measures), the input data may be resampled, by simulation of optimization input parameters in a manner statistically consistent with the first set of data, as described, for example, by J. Jobson and B. Korkie, "Estimation for Markowitz Efficient Portfolios," *Journal of Portfolio Management*, (1981), which is herein incorporated by reference. Embodiments of the present invention are related to improvements and extensions of the teaching of the Michaud '018 patent.

When portfolios are rebalanced in accordance with current practice, criteria are applied that are typically not portfolio-based or consistent with principles of modern statistics but are generally associated with various ad hoc rules. U.S. Pat. No. 6,003,018 teaches a portfolio-based rebalancing criterion that can be used for all portfolios on the resampled efficient frontier and that is consistent with principles of modern statistics and considers the uncertainty in investment information.

Michaud 1998, provided data, for purposes of illustration, that consisted of 18 years of monthly returns for eight asset classes. The resampling process illustrated in the text computes simulated efficient frontiers of 18 years of returns, or 216 monthly resampled returns for each set of simulated means and covariances and associated simulated efficient frontiers, prior to obtaining the average. In this instance the resampling of returns duplicates the amount of information in the historical return dataset. It is desirable to allow for other variable assessments of confidence in the forecasting power of the data, and that is addressed, below, in the context of the present invention.

Other features of rebalancing procedures, as practiced heretofore, imposed important limitations. The discriminatory power was not customizable, with too high power at low levels of risk and too little power at high levels of risk. Methods are clearly necessary for providing relatively uniform discriminatory power across portfolio risk levels as well as being able to customize discriminatory power according to the investment needs of organizations which differ in terms of user sophistication, asset class characteristics, or investment strategy requirements. Methods are also clearly desirable that help identify anomalously weighted assets (overly large or small weights) relative to a normal range that is associated with the uncertainty of investment information.

SUMMARY OF THE INVENTION

In accordance with preferred embodiments of the present invention, a computer-implemented method is provided for selecting a value of a portfolio weight for each of a plurality of assets of an optimal portfolio. The value of portfolio weight is chosen from specified values associated with each asset, between real numbers c_1 and c_2 that may vary by asset, each asset having a defined expected return and a defined standard deviation of return. Each asset also has a covariance with respect to each of every other asset of the plurality of assets. The method has steps of:

a. computing a mean-variance efficient frontier based at least on input data characterizing the defined expected return and the defined standard deviation of return of each of the plurality of assets;

b. indexing a set of portfolios located on the mean-variance efficient frontier thereby creating an indexed set of portfolios;

c. choosing a forecast certainty level for defining a resampling process of the input data consistent with an assumed forecast certainty of the input data;

d. resampling, in accordance with the process defined by the forecast certainty level, a plurality of simulations of input data statistically consistent with the defined expected return and the defined standard deviation of return of each of the plurality of assets;

e. computing a simulated mean-variance efficient portfolio for each of the plurality of simulations of input data;

f. associating each simulated mean-variance efficient portfolio with a specified portfolio of the indexed set of portfolios for creating a set of identical-index-associated mean-variance efficient portfolios;

g. establishing a statistical mean for each set of identical-index-associated mean-variance efficient portfolios, thereby generating a plurality of statistical means, the plurality of statistical means defining a resampled efficient frontier,

wherein processes (a), (b), and (d)-(g) are digital computer processes;

h. selecting a portfolio weight for each asset from the resampled efficient frontier according to a specified utility objective; and

i. investing funds in accordance with the selected portfolio weights.

In accordance with other embodiments of the invention, the specified utility objective may be a risk objective, and steps (d) through (i) may be repeated based upon a change in a choice of forecast certainty level. The forecast certainty level may be based upon investment horizon or other factors.

In yet other embodiments of the invention, the step of indexing the set of portfolios may include associating a rank with each portfolio of the indexed set of portfolios located on the mean-variance efficient frontier, and may also include associating a parameter for which a specified measure of expected utility is maximized, one possible specific measure of expected utility represented by a quantity $\mu-\lambda\sigma^2$, where σ^2 is the variance of each portfolio and μ is the defined expected return of each portfolio of the set of portfolios located on the mean-variance efficient frontier.

In accordance with another aspect of the present invention, computer-implemented methods are provided for selecting a value of a portfolio weight for each of a plurality of assets of an optimal portfolio. In these embodiments, the methods have steps of:

a. computing a mean-variance efficient frontier, the frontier having at least one mean-variance efficient portfolio, based at least on input data characterizing the defined expected return and the defined standard deviation of return of each of the plurality of assets;

b. choosing a forecast certainty level from a collection of forecast certainty levels for defining a resampling process of the input data consistent with the assumed forecast certainty of the input data;

c. generating a plurality of optimization inputs drawn at least from a distribution of simulated optimization inputs consistent with the defined expected return, the defined standard deviation of return of each of the plurality of assets and the chosen forecast certainty level;

d. computing a simulated mean-variance efficient frontier, the frontier having at least one simulated mean-variance efficient portfolio, for each of the plurality of optimization inputs;

e. associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios for creating a set of associated mean-variance efficient portfolios;

f. establishing a statistical mean for each set of associated mean-variance efficient portfolios, thereby generating a plurality of statistical means, the plurality of statistical means defining a meta-resampled efficient frontier;

g. selecting a portfolio weight for each asset from the meta-resampled efficient frontier according to a specified investment objective; and

h. investing funds in accordance with the specified portfolio weights.

In various alternate embodiments of the invention, the specified investment objective may be a risk objective. The step of choosing a forecast certainty level from a collection of forecast certainty levels may include choosing from a set of indices from 1 to N, where each index is calibrated to provide a different level of forecast certainty. The number of samples drawn from a distribution may be increased based on a higher level of forecast certainty. Each level of forecast certainty may represent a geometric increase (or decrease) in the number of observations drawn from the distribution.

In further alternate embodiments of the invention, the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios may include associating on the basis of proximity to a maximized expected utility, where, for a specific example, the expected utility may be a quantity $\mu-\lambda\sigma^2$.

The step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios may include associating on the basis of a measure of risk, where the measure of risk may be variance or residual risk.

In accordance with yet further embodiments of the invention, the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios may include associating on the basis of a measure of return such as expected return or expected residual return. The step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios may also include associating on a basis of proximity of the portfolios to other simulated mean-variance efficient portfolios or associating on a basis of ranking by a specified criterion, the criterion chosen from a group including risk and expected return. Alternatively, the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios may include associating on a basis of investment relevance of the simulated mean-variance efficient portfolios or associating simulated portfolios based on a selection of relevant simulations.

The step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios may include specifying a fraction of portfolios per simulation to be considered in the association.

Additional steps of the method may include calculating an estimated normal range of asset weights for an optimal portfolio and calculating a portfolio rebalancing probability for a given portfolio with respect to an optimal portfolio.

In yet another aspect of the present invention, a computer program product is provided for use on a computer system for selecting a value of portfolio weight for each of a specified plurality of assets of an optimal portfolio. The computer program product has a computer usable medium having computer readable program code thereon, with the computer readable program code including:

a. program code for computing a mean-variance efficient frontier based at least on input data characterizing the defined expected return and the defined standard deviation of return of each of the plurality of assets;

b. program code for aggregating a set of portfolios located on the mean variance efficient frontier;

c. a routine for generating a plurality of optimization inputs drawn from a distribution of simulated optimization inputs statistically consistent with the defined expected return and the defined standard deviation of return of each of the plurality of assets;

d. program code for computing a simulated mean-variance efficient frontier, the frontier comprising at least one simulated mean-variance efficient portfolio for each of the plurality of optimization inputs;

e. program code for associating each simulated mean-variance efficient portfolio with a specified portfolio of the set of aggregated portfolios for creating a set of associated mean-variance efficient portfolios;

f. a module for establishing a statistical mean for each set of associated mean-variance efficient portfolios, the plurality of statistical means defining a meta-resampled efficient frontier; and

g. program code for selecting a portfolio weight for each asset from the meta-resampled efficient frontier according to a specified risk objective.

In alternative embodiments, the routine for resampling a plurality of simulations on input data may draw a number of returns from a simulation based on a specified level of estimation certainty, and the set of associated mean-variance efficient portfolios may be based on a specified level of forecast certainty.

BRIEF DESCRIPTION OF THE DRAWINGS

The invention will be more readily understood by reference to the following description, taken with the accompanying drawings, in which:

FIG. **1**A depicts the prior art principle of calculating an efficient frontier of maximum expected return for each given level of portfolio risk;

FIG. **1**B depicts the prior art principle of calculating an efficient frontier of maximum expected residual return for each given level of portfolio residual risk;

FIG. **2** displays a set of statistically equivalent portfolios within the risk/return plane;

FIGS. **3**A-**3**F represent constituent modules of a flow-chart depicting a process for computing a portfolio rebalancing index, in accordance with embodiments of the present invention;

FIG. **4** displays statistically equivalent portfolios within the risk/return plane corresponding to three particular risk rankings on the efficient frontier: minimum variance, maximum return, and a middle return portfolio;

FIG. **5** shows the resampled efficient frontier plotted in the risk/return plane in accordance with a preferred embodiment of the present invention;

FIG. **6** shows the classical and several resampled efficient frontiers, illustrating various levels of estimation certainty in accordance with embodiments of the present invention;

FIG. **7** is a plot showing an identified portfolio on a resampled efficient frontier based on a specified maximum expected utility point;

FIG. **8**A is a plot showing a typical need-to-trade probability for a specified current portfolio relative to a specified target portfolio as a function of a specified portfolio risk;

FIG. **8**B is a plot showing need-to-trade probabilities for a three specified current portfolios relative to corresponding specified target portfolios as a function of a specified portfolio risk, using classical efficient portfolios; and

FIG. **8**C is a plot showing need-to-trade probabilities for a three specified current portfolios relative to corresponding specified target portfolios as a function of a specified portfolio risk, using resampled efficient portfolios in accordance with embodiments of the present invention.

DETAILED DESCRIPTION OF SPECIFIC EMBODIMENTS

MV-efficient portfolios may be recalculated, based on distinct resamplings of data, subject to the same constraints as applied in the initial solution. Inputs to each solution include the forecast returns and standard deviations and distributional assumptions of returns drawn, typically, for a plurality of asset classes (stocks, mutual funds, currencies, etc.) from a multivariate distribution. Correlations across asset classes are also input.

In accordance with preferred methods for resampling when no additional distributional information is known, simulated returns are drawn for a plurality of asset classes from a multivariate normal distribution. Other resampling methodologies may entail bootstrapping of historical data or resampling under distributional assumptions (for instance, if skew or kurtosis of returns is known). This Monte Carlo simulation generates different forecasts of risk and return, including, but not limited to, mean, standard deviation, and correlation, to be used in each computation of MV-efficient portfolios.

As an alternative procedure when the distribution of simulated optimization inputs is known, resampled optimization input parameters (mean return, standard deviation, correlations, etc.) may be drawn from their distribution. For example, if multivariate normality of returns is assumed and the vector of returns r for all assets is modeled by $r \sim N(\mu,\Sigma)$, and a forecast certainty corresponding to N observations of return is given, in each simulation i, a forecast mean return μ_i and covariance of returns Σ_i can be modeled by:

$$\mu_i \sim N(\mu,\Sigma/N)$$

$$\Sigma_i \sim \text{Inv-Wishart}_{N-1}(\Sigma),$$

where Wishart distributions, their notation, and their application to unknown covariance matrices are discussed in detail in Gelman et al., *Bayesian Data Analysis*, Chapman & Hall (1995), which is incorporated herein by reference.

The constraints applied to each of the solutions, given the aforesaid optimization inputs, include constraints placed on the set of weights accorded to the various components of the portfolio. An efficient solution may be sought in which the solution is a set of weights $\{w_i\}$ of assets comprising a portfolio or, else, a set of active weights $\{x_i\}$ of assets defined differentially with respect to a benchmark portfolio. In the latter case, the benchmark portfolio is designated, in FIG. **1**B, as point 'A' with respect to which residual risk and expected residual return are plotted in the figure. In either case (i.e., whether the weights are to be solved for absolutely or with respect to a benchmark portfolio), the weights are subject to constraints specified by the user. These constraints may include, for example, a constraint that one or more specified weights, or the sum of specific weights, must lie between two specified real numbers, c_1 and c_2. For example, c_1 and c_2 may correspond, respectively, to 0 and 1. The inclusion of negative weights allows the resampled frontier, discussed below, to include portfolios of both long and short asset weights. Similarly, the sum of portfolio weights may be constrained, for example, by requiring the sum to equal an amount to be invested, or, conventionally, 1.

Based on multiple resamplings, as shown in FIG. **2**, a set **12** of statistically equivalent MV efficient portfolios may be calculated. By iterating this procedure, a large MV efficient "statistical equivalence" set of portfolios, in the expected return—portfolio variance space, may be generated. Multiple resamplings may be based upon returns drawn a specified number of times from an assumed distribution for a particular asset class. All a priori sets of assumptions with respect to the distribution are within the scope of the present invention; for example, the distributions may be defined by bootstrapping, normal, log-normal, mixed, etc. As an alternative algorithm, statistical input parameters (mean return, standard deviation, correlations) may be derived from a set of returns drawn from a particular simulation, and those parameters, in turn, used, in a bootstrapping manner, for subsequent resamplings, thus resampled distributions may serve the same function as resampled returns with respect to derivation of a resampled efficient frontier that may be referred to as a 'meta-resampled efficient frontier.'

Referring now to FIGS. **3**A-**3**F, a process is described for deriving a Portfolio Rebalancing Index, which describes the percentile level of a portfolio norm relative to a distribution of portfolio norms, as will be described below.

As shown in FIG. **3**A, a set of initial Optimization Inputs, based on historical performance data relative to a set of assets, or otherwise, is provided by a user. Based on any such Optimization Inputs, a set of K mean-variance (MV) optimized portfolios may be calculated using classical Markowitz optimization. These K mean-variance optimized portfolios span the mean-variance efficient frontier for a given set of optimization inputs, in that they encompass the range either from minimum low to maximum high risk or minimum low to maximum high expected return. Any one of these portfolios may be referred to herein, and in any appended claims, as optimal portfolios, understanding that the optimization criterion might be based upon any specified utility function and not just one having a preference for mean variance alone. Optimization criteria may be based on specified investment objectives such as portfolio risk or a specified utility function. The set of K portfolios spanning the mean-variance efficient frontier for a given set of optimization inputs will be referred to as 'K-MV'.

At this stage, three parameters may be defined, which will have the respective significance as now indicated:

N designates the total number of resampled sets of optimization inputs of the initial optimization inputs;

M designates the number of resampled sets of optimization inputs of a given set of resampled optimization inputs; and

M**2** designates the number of randomly chosen K-MV portfolios from the N simulated K-MV portfolios.

The running index i is the index over which successive variables are stepped for successive sets of resampled optimization inputs in deriving a distribution of portfolio norms.

Until the process is complete, i.e., as long as i is less than N, as i is incremented, a set of resampled inputs is created in the step labeled Resample Inputs. In this step, a new set of risk and return inputs is created for a given level of confidence in the inputs. The distribution for the resampled inputs is based on the original inputs (or the previous resampled inputs when resampling inputs in module B, as described below). An efficient frontier (designated EFF(i)) is derived for each new set of resampled inputs, namely the set K-MV described above, and the same set is also stored as a set of Equivalent Portfolios EQ(i).

Module B (FIG. **3**B) performs N successive resamplings and stores the K-MV efficient frontier portfolios in the array EFF. Once this has been completed, Module C (FIG. **3**C) first associates the portfolios of the simulated K-MV with portfolios from other K-MV efficient portfolios. Various procedures for associating portfolios of K-MV sets are discussed below, whether by rank order value from each K-MV, or otherwise. Once portfolios from the various K-MVs have been associated, they are averaged (again, in a generalized sense, as described below) in order to derive the resampled efficient frontier (designated REF).

As shown in FIG. **3**D, if M**2** is non-zero, associated portfolios are bootstrapped, that is to say, M**2** associated portfolios are randomly selected from the set of simulated equivalent portfolios and averaged to form a new set of equivalent portfolios EQ(i).

Once the new set of equivalent portfolios is set up, statistics may be performed on the asset weights of the K portfolios of the Resampled Equivalent Frontier with respect to the distribution of the associated asset weights, including their respective confidence percentiles, and other statistical measures, as shown in FIG. **3**D.

Referring, finally, to FIG. **3**F, portfolio norms are computed, as discussed further below, for each simulated portfolio with respect to every portfolio (or a relevant subset) of the resampled efficient frontier. The various norms that may be employed are discussed below, but one norm entails the tracking error between two specified portfolios. In order to gauge the advisability of a portfolio rebalancing, for all K portfolios of the resampled efficient frontier, a set of simulated portfolios is chosen from EQ(i) that represents those deemed to be the most reasonable alternative investments. In a preferred embodiment of the invention, some fraction, typically about 5%, of the portfolios are selected, where this fraction encompasses those simulations with the smallest portfolio norm with respect to each corresponding portfolio of the resampled efficient frontier. The portfolio rebalancing index may then be determined as the percentile level of portfolio norm for a specified portfolio with respect to the portfolio norm distribution.

Rather than resampling based on historical returns, a further alternative procedure entails factor model resampling, wherein the set of returns is modeled according to a linear (in this example) model, such as:

$$r_i=\alpha_i+\beta_i'F+(\text{any other mod eled terms})+\epsilon_i,$$

where, for each asset 'i', α signifies asset-specific excess return, F signifies the vector of factor returns for all assets (be they returns associated with a representative portfolio, stylized factor, etc.), β'_i signifies a (transposed) vector of coefficients to any factor return for asset 'i' (be they estimated historical, or, alternatively, current characteristic values of the asset, e.g., dividend-to-price ratio), and is a residual stochastic component of the return specific to asset 'i'. Factor returns may be modeled in a variety of ways, but typically each represents pervasive market factors, and, in this model, represents any variable influencing return of two or more assets. Statistical inputs may be specified for distributions of each of the constituent terms, whether based on systematic or idiosyncratic variance, which constituent terms may accordingly then be resampled.

Based on application of a specified statistical procedure, an existing portfolio (or, the "CURRENT portfolio") may be found to be consistent, in a statistical sense, with efficiency and thus not require optimization, thereby potentially and advantageously saving transaction costs associated with revision of a portfolio. Alternatively, based on the same statistical procedure, an existing portfolio may be found to be inconsistent with efficiency and an alert produced indicating a need to rebalance to the appropriate target optimal portfolio. Implementation of a rebalancing test is described in detail below. Referring to FIG. **4**, statistically equivalent portfolios within the risk/return plane are shown corresponding to three particular risk rankings on the efficient frontier: namely, minimum variance **14**, maximum return **16**, and a middle return portfolios **18**.

In the context of statistical equivalence of portfolios, a "when-to-trade probability" or "rebalance probability" is defined as the confidence level of rejection test for portfolio statistical equivalence. More particularly, with respect to an "optimal resampled efficient portfolio," described in detail below, the "when-to-trade probability" is defined as the percentage of resampled portfolios "closer" (in terms of a norm to be discussed below) to the "optimal resampled efficient portfolio" than the portfolio in question, i.e., the CURRENT portfolio. The norm typically employed is that of the variance of a portfolio with respect to a particular optimal resampled efficient portfolio (commonly referred to as tracking error), however other norms are within the scope of the present invention.

Referring further to FIG. **4**, all resampled portfolios within the risk/return plane may be associated, many-to-one, with particular portfolios on MV efficient frontier **10**. Various criteria may be applied in associating portfolios with those on the MV efficient frontier, and all such associations are within the scope of the present invention. As one example, each of the K efficient frontier portfolios (i.e., each point on efficient frontier **10**) may be identified by its relative return rank. Similarly, the efficient frontier portfolios may be ranked by their variance, the maximum variance corresponding to the maximum return, the rankings by risk or return similarly mapping onto one another uniquely. Thus, for example, the minimum variance portfolio **14** might have the lowest rank relative to the other efficient portfolios of efficiency frontier **10**. Similarly, maximum average return portfolio **16** has the highest average return rank in each simulated efficient frontier. Similarly, any other simulated portfolio is rank associated with a particular efficient frontier portfolio. The sparsely clustered portfolios **18** shown in the figure correspond to the 'middle' ranked efficient portfolio. In practice, the shape of the rank-associated regions varies in dependence upon the position of the portfolio on the MV efficient frontier.

It is not necessary, however, that the association with efficient frontier portfolios be by rank, and particular portfolios on the MV efficient frontier may be indexed, and thus index-associated, each with a set of statistically equivalent efficient portfolios lying below the efficient frontier. Indexing, for example, of the set of MV efficient portfolios may be by associating with each MV efficient portfolio a "lambda value," defining the risk/return preference, with respect to which the quantity $\phi=\sigma^2-\lambda\mu$ is minimized, where σ^2 is the variance of each portfolio and μ is the expected return of each portfolio of the set of portfolios located on the mean variance efficient frontier. The parameter λ assumes a value between zero and infinity. The foregoing procedure is mathematically equivalent to maximizing the quantity $\mu-\lambda\sigma^2$.

Moreover, association of efficient portfolios for deriving an average (in a specified sense), and thus a resampled efficient frontier, are not required to be index-ranked at all, within the scope of the present invention. Indeed, in alternate embodiments of the invention, 'neighboring', or otherwise related, portfolios may be grouped, in order to achieve desired aggregation of portfolio characteristics.

Methods for association for averaging portfolios relative to an index set include equal or weighted averages by significance or other nearness measures where the index set defined for all simulated portfolios in the resampling process. An example of an index set may include the set of simulated portfolios with similar utility values. In a further example, a resampled frontier may represent a best-fit curve through portfolio space of simulated portfolios with weighted averages taken over some specified subset.

An alternative means of associating portfolios from distinct ensembles of portfolios, say, for example, as derived from successive resamplings, may be on the basis of maximizing expected utility. Referring to FIG. **7**, an efficient frontier **60** is shown in the risk-return plane. Various utility functions **62**, **64**, and **65** are plotted as may be specified for particular investors. Each plotted utility function curve, say **64** for example, represents a constant utility for the specified investor. Utility function **62**, for example, obeys a different functional law from that of utility function **64**. Functional forms typically employed may be exponential and/or polynomial functions of risk.

Utility functions **64** and **65** obey an identical functional dependence of return vs. acceptable risk, differing only in the total utility, indicated, by way of example, by the quantity u which assumes the value **5** and **6** for curves **64** and **65** respectively. The point **68** on efficient frontier **60** may be characterized in that it represents a portfolio that maximizes the expected utility with respect to the class of utility functions to which curves **64** and **65** belong. Similarly, point **66** on efficient frontier **60** may be characterized in that it represents a portfolio that maximizes the expected utility with respect to the class of utility functions to which curve **62** belongs. Other points on efficient frontier **60** may be characterized in terms of their 'distance' (as defined by a specified norm) from point **68** of maximum expected utility. Similarly, points on different efficient frontiers may be associated, as described herein in the context of defining a resampled efficient frontier, on the basis of their identity or proximity to portfolios maximizing expected utility with respect to specified utility functions.

Rebalancing of a portfolio is indicated if the current portfolio is statistically distinct from a target optimized portfolio on a resampled efficient frontier identified according to criteria to be discussed below. "Proximity" of one portfolio (whether resampled, indexed or otherwise) to a corresponding portfolio may be defined in terms of a test metric based on a "norm," with the norm having the usual properties of a distance function as known to persons skilled in the mathematical arts. The properties of a norm defined in a vector space are well-known: a norm associates a non-negative value with any vector in the space, preserves scalar multiplication, and obeys the relation $\|x+y\| \leqq \|x\|+\|y\|$.

Various norms may be used for defining distance in the risk/return space. The distance criterion for any portfolio P is typically taken to be the relative variance for portfolio P,

$$(P-P_0)^t*S*(P-P_0),$$

where $P-P_0$ is the difference vector of portfolio weights with respect to P_0, the corresponding index-associated portfolio on the resampled efficient frontier, and, S is the input return covariance matrix (with the superscript 't' denoting the transpose of the difference vector). The norm is taken in the space of portfolio vectors (i.e., "portfolio space"). Alternative distance criteria may include additional functions (linear or otherwise) of $P-P_0$.

The CURRENT portfolio is measured for statistical distinction against a set of simulated portfolios, the enhanced discriminatory 'power' of the balancing test reflecting the decreasing likelihood that the desirability of rebalancing a portfolio is 'missed.' I.e., a more powerful rebalancing test is less likely to attribute a CURRENT portfolio to a distribution of like portfolios where it properly does not belong to the population. Alternatively, controlling discriminatory power enhances the likelihood of rebalancing only where appropriate.

Referring to FIG. **8**A, an example is shown in which the need-to-trade probability for a given CURRENT portfolio, as determined by a particular rebalancing test, is plotted as a function of portfolio risk. The plot of FIG. **8**A thus reflects the enhanced discriminatory power of the specified rebalancing test. The CURRENT portfolio may have an associated residual risk close to a specified value σ_A, thereby lying sufficiently close to a specified efficient frontier as not to be statistically distinguishable from a target portfolio. The need-to-trade probability may be close to zero over some range, whereas, outside that range, the probability of a given test dictating a need to rebalance rises in case it is desirable to achieve a different level of risk.

The portfolios against which a CURRENT portfolio is tested are typically weighted in performing a statistical rebalancing test. In accordance with embodiments of the present invention, a set of portfolios is retained for performing the statistical test of the CURRENT portfolio with the objective of increasing the discriminatory power of the statistical test. An advantageous benefit that this procedure may provide is that of substantial uniformity of power of the rebalancing test across the entire efficient frontier. This may advantageously be achieved by weighting sample portfolios increasingly with proximity to a target portfolio. Additionally, this procedure may be used advantageously to reduce computational overhead in performing a rebalancing test.

Once an ensemble of MV efficient portfolios has been associated, whether by index-set association, proximity to a maximum expected utility point, or other method of aggregation, usual statistical measures of the ensemble may be derived. These measures include, without limitation, the averages, standard errors, and t-statistics of the average of the portfolio weights of the rank-associated simulated efficient portfolios. Referring now to FIG. **5**, an average of index-associated MV efficient portfolios may be defined, in accordance with preferred embodiments of the present invention, the average of index-associated MV efficient portfolios being referred to as a "resampled-efficient portfolio." The average may be determined with respect to any of a variety of parameters, and, in accordance with a preferred embodiment, it is with respect to the vector average of the associated portfolios. The vector average of a set of portfolios is defined as the average over the weighted assets of each of the portfolios of the set, taking into account the sign, positive or negative, of the contribution of a particular asset to a particular portfolio. The locus **40** of resampled-efficient portfolios is referred to as the "resampled efficient frontier." The resampled-efficient portfolio and its associated statistics may be applied as a statistical measure for portfolio analysis, as further described herein. Its application, as a choice for portfolio selection, advantageously removes, by definition, the "outlier" portfolios which strongly depend on values of a particular set of inputs and improves out-of-sample performance, on average. Statistical procedures for portfolio analysis and revision, and performance benefits based on these concepts, are further described in detail in the Michaud book.

Forecast Certainty Levels

In particular, the number of resampling simulations that are performed and efficient frontiers that are computed may be specified, prior to obtaining the average, or resampled efficient frontier, as discussed in the foregoing paragraph. The number of resamplings of returns performed to compute simulated means and covariances and associated simulated MV efficient frontiers prior to obtaining the average, or resampled efficient frontier, is a free parameter of the resampled efficiency optimization procedure.

As discussed in the Background Section, above, Michaud 1998 taught resampling of returns that duplicated the amount of information or forecast certainty in the historical return dataset. In the same way, for any historical return data set of assets, the implicit information or forecast certainty of the dataset can be replicated by setting the number of resamplings or bootstraps of returns to equal the number of historical returns in the data set.

In practice, the implicit information or forecast certainty of a historical return dataset may not be appropriate. For example, the information level of a 100 year monthly historical return dataset may be inconsistent with computing an optimal portfolio for the following month. Alternatively, the means and covariances used to compute an optimal portfolio is often not based on historical return data. In that case the number of resamplings of returns to compute simulated efficient frontiers prior to obtaining the average or resampled efficient frontier must be found from other considerations. In addition, the information level associated with the means and covariances used to compute an optimal portfolio or efficient frontier will often vary depending on time period, outlook, and many other considerations even when the inputs are unchanged.

The number of resamplings of returns that are performed for computing simulated means and covariances and associated efficient frontiers is a natural framework for modeling forecast certainty in the optimization process. As the number of resamplings increases without limit, the simulated means and covariances and associated efficient frontier approaches the classical MV efficient frontier and the average or resampled efficient frontier approaches the classical Markowitz MV efficient frontier. The limit is the case of complete certainty in the optimization inputs and the classical MV efficient frontier.

Referring now to FIG. **6**, higher numbers of returns drawn from a simulation correspond to higher levels of estimation certainty. In FIG. **6**, the uppermost efficient frontier **10** is, as before, the classical efficient frontier, determined on the basis of the original inputs associated with each asset. Resampled efficient frontiers **50**, **52**, **54**, **56** and **58**, correspond to increasing numbers of simulations (or numbers of simulations periods per simulation), and thus to increasing levels of estimation certainty.

Conversely, as the number of resamplings of returns decreases, the simulated means and covariances and efficient frontiers will vary widely and the average or resampled efficient frontier will increasingly resemble the no-information equal weighted or benchmark weighted portfolio. The limit is the case of complete uncertainty and equal or benchmark portfolio weighting.

More particularly, in accordance with embodiments of the present invention, forecast certainty may also vary among input data sets, with particular data sets giving rise to greater forecast certainty in predicted performance than other input data sets. Optimization inputs generally reflect a forecast or expected return process assumed to have, on average, a positive correlation with ex post returns. The level of information is typically expressed as the "information correlation" (IC)—e.g., an IC of 0.2 reflects an expected correlation of 0.2 of forecast with ex post returns.

Information correlation refers to the assumed correlation between a forecast and ex post actual return. IC typically varies with firm strategy, industry, sector, etc. and may be stated, generally, to be a proxy for relative forecast certainty associated with a particular data set. More generally, forecast certainty may be associated with the level of information correlation (IC) and also by the standard deviation of the IC distribution

Forecast certainty level is defined with respect to the number of resampling returns used to compute the simulated means and covariances and associated efficient frontiers in the resampled efficient frontier process. Forecast certainty level shows that the resampled efficient frontier is a generalization of classical mean-variance efficiency that allows the manager to control the amount of confidence in the inputs in the optimization process. At one extreme of confidence, resampled efficiency optimization is Markowitz efficiency; at the other extreme of confidence resampled efficiency results in either equal weighting or, in the presence a benchmark index, the benchmark portfolio.

One purpose of the resampling procedure is to include an appropriate level of forecast uncertainty into the resampled optimization inputs. To that end, resampled inputs may be designed to reflect a range of forecast certainty levels. The resampling parameter N provides a means of controlling the level of certainty implicit in the forecast process.

In accordance with preferred embodiments of the present invention, an index value, assuming a numerical value between 1 and N, is associated with a forecast certainty attributed by an analyst to a data set. The index value corresponds to the number of resamplings for a particular simulation. Thus, the more resamplings, the higher the forecast certainty, with higher index values corresponding to successive resampled efficient frontiers, **50**, **52**, **54**, **56** and **58**, as depicted in FIG. **6**.

In a preferred embodiment, ten levels of certainty may be defined, where 1 represents very uncertain and ten represents high but not perfect forecast certainty. In this preferred embodiment, an "average" level of forecast certainty is chosen for a particular application. For example, for equity portfolio optimization, level 4 may be defined as corresponding to a level of certainty consistent with the way many commercial risk models are estimated. In this case level 4 may be defined as drawing five years or sixty months of monthly observations from the monthly distribution. Other certainty levels associated with each level represent a geometric increase (or decrease) in the number of observations drawn from the distribution (e.g. if level 4 corresponds to 60 monthly observations, level 5 may correspond to 90 monthly observations and level 6 to 135 monthly observations).

The forecast certainty level assumption impacts all the resampled efficient frontier computations. Forecast certainty level defines the resampled efficient frontier and optimized portfolios, the resampled need-to-trade probability and asset weight ranges. Forecast certainty level may vary over time to dynamically control the optimized portfolio or resampled efficient frontier, need-to-trade probability, and asset range monitoring rules.

The way a manager chooses a level of forecast certainty is subjective and context dependent. The level may be associated with the time horizon of the optimization. For example, an optimal portfolio that is appropriate for a long-term investment strategy is likely to have a different forecast certainty level than one for short-term active management. Intuitively, it is more likely that stocks will beat bonds over a ten year horizon than for any given month. Different strategies, information sets, and different time periods will drive the choice of forecast certainty level. But some choice of forecast certainty level and resampled efficiency optimization is almost always advisable. This is because no investor, in practice, is ever 100% certain of their information. MV optimization does not include any sense of estimation error and essentially requires a metaphysical forecast certainty level for application.

Refined Discrimination Power of Rebalancing Tests

It is a characteristic of classical efficient frontiers that portfolios at the high-risk extreme tend to be concentrated—specifically, with high weighting of assets bearing high expected returns. On the other hand, portfolios of resampled efficient frontiers at the corresponding high-risk extreme tend to be substantially diversified.

FIG. **8**B shows that this leads to unsatisfactory statistical implications if a target portfolio based on resampled optimizations is compared with portfolios on a classical efficient frontier. In particular, for the case of low-risk portfolios, points plotted along curve **72** show a need-to-trade (i.e., a probability near unity) in almost all instances, where classical efficient portfolios are employed, since optimization will emphasize a particular low-risk target asset. Similarly, for the case of high-risk portfolios, classical efficient portfolios will also indicate a need-to-trade in most cases, as shown along curve **76**, in this case because maximum-return assets will be emphasized. A convenient method for calibrating the portfolio rebalancing procedure is to select a set of portfolios that are considered to be statistically equivalent to an optimal portfolio and adjust the free parameters so as to include the set of portfolios in the statistically equivalent region for an appropriate acceptance level. Curve **74** depicts the case of medium-risk portfolios indicating little discriminatory power with respect to high risk efficient portfolios.

FIG. **8**C shows that need-to-trade probabilities are ameliorated at both the low-risk (curve **82**) and high-risk (curve **86**) ends, as well as for the case of medium risk (curve **84**), where resampled efficient frontiers are used for discrimination of need-to-trade, in accordance with the present invention. In each instance, a region of nearly symmetric 'tolerance' exists where reasonably diversified portfolios are determined to be statistically similar to optimized targets.

Consequently, in accordance with preferred embodiments of the invention, an associated meta-resampled efficient frontier is found, as described above, for each classical efficient frontier corresponding to a simulation of returns. This procedure may advantageously lead to a rebalancing test of greater uniformity, particularly at the high-risk end of the resampled efficient frontier, and thereby advantageously providing for automation of some or all rebalancing decisions if so desired.

Additionally, in certain circumstances, it is desirable to control the power of a rebalancing test across the spectrum of portfolio risk in accordance with particular investment objectives and strategies, objectives, and end-user requirements. One means to that end is ensuring the statistical relevance of the ensemble of statistically weighted portfolios against which a CURRENT portfolio is tested. To that end, two parameters are significant: the percentage of relevant simulations to keep for purposes of statistical comparison with the CURRENT portfolio, and a 'relevance' criterion governing the number of portfolios retained within the ensemble per simulation. The first parameter, generally, has the effect of increasing the power of a rebalancing test as fewer simulations are kept. With respect to the second parameter, increasing the number of portfolios per simulation reduces the power of the test while, at the same time, typically spreading out the power over an increased range of portfolio risk.

Restriction of the percentage of simulations, or of the considered portfolios per simulation, for statistical consideration may be further refined by averaging simulations to yield meta-resampled efficient frontiers, thereby deriving benefits associated with comparing resampled portfolios with resampled, rather than classically optimized, portfolios, as described above.

Estimating Normal Ranges of Portfolio Weights

Defining an investment-relevant normal range of portfolio weights provides useful guidelines for many asset management functions including at-a-glance identification of anomalous portfolio structure and instances in which portfolio rebalancing to optimality may be advisable.

For each asset of a target optimal portfolio, an enhanced estimate of the normal range of asset weights can be calculated. The range incorporates the distribution of the asset weights of the associated meta-resampled or bootstrapped resampled portfolios. In a preferred embodiment, the associated meta-resampled portfolios asset weights for each asset of the target portfolio form a meta-resampled distribution of asset weights. Various descriptive statistical measures can then be applied to provide an estimate of a normal range relative to each asset of the target optimal portfolio. For example, in the table below, the 25th and 75th percentile values of the meta-resampled distribution of asset weights for each asset for ten asset classes for medium and high risk resampled efficient target portfolios based on historical return data are given. Unlike current asset range estimate methods, resampled methods provide investment relevant estimates that vary in statistical characteristics by asset and target portfolio risk.

Resampled Index-Associated Portfolio Weights Range Estimates

	Medium Risk		High Risk	
	25th Pctile	75th Pctile	25th Pctile	75th Pctile
Money Market	10%	22%	0%	0%
Intermediate Fixed	12%	34%	0%	1%
Long Term Fixed	3%	14%	1%	12%
High Yield Corp	3%	19%	0%	4%
Large Cap Value	2%	13%	1%	17%
Large Cap Growth	0%	5%	1%	12%
Small/Mid Cap Value	0%	3%	0%	7%
Small/Mid Cap Growth	0%	4%	4%	35%
International Stocks	2%	14%	3%	30%
Real Estate	3%	15%	2%	25%

In alternative embodiments, the disclosed methods for evaluating an existing or putative portfolio may be implemented as a computer program product for use with a computer system. Such implementations may include a series of computer instructions fixed either on a tangible medium, such as a computer readable medium (e.g., a diskette, CD-ROM, ROM, or fixed disk) or transmittable to a computer system, via a modem or other interface device, such as a communications adapter connected to a network over a medium. The medium may be either a tangible medium (e.g., optical or analog communications lines) or a medium implemented with wireless techniques (e.g., microwave, infrared or other transmission techniques). The series of computer instructions embodies all or part of the functionality previously described herein with respect to the system. Those skilled in the art should appreciate that such computer instructions can be written in a number of programming languages for use with many computer architectures or operating systems. Furthermore, such instructions may be stored in any memory device, such as semiconductor, magnetic, optical or other memory devices, and may be transmitted using any communications technology, such as optical, infrared, microwave, or other transmission technologies. It is expected that such a computer program product may be distributed as a removable medium with accompanying printed or electronic documentation (e.g., shrink wrapped software), preloaded with a computer system (e.g., on system ROM or fixed disk), or distributed from a server or electronic bulletin board over the network (e.g., the Internet or World Wide Web). Of course, some embodiments of the invention may be implemented as a combination of both software (e.g., a computer program product) and hardware. Still other embodiments of the invention are implemented as entirely hardware, or entirely software (e.g., a computer program product).

Once an optimized set of portfolio weights is determined according to a specified risk objective and in accordance with the foregoing teachings, funds are invested in accordance with the portfolio weights that have been determined.

The described embodiments of the invention are intended to be merely exemplary and numerous variations and modifications will be apparent to those skilled in the art. All such variations and modifications are intended to be within the scope of the present invention as defined in the appended claims.

What is claimed is:

1. A computer-implemented method for selecting a value of a portfolio weight for each of a plurality of assets of an optimal portfolio, the value of portfolio weight chosen from specified values between real numbers c_1 and c_2 associated with each asset, each asset characterized by an expected return and a standard deviation of return, and a covariance with respect to each of every other asset of the plurality of assets, the method comprising:

a. choosing a forecast certainty level from a collection of forecast certainty levels for defining a resampling process of the input data consistent with the assumed forecast certainty of input data characterizing the expected return and standard deviation of return of each of the plurality of assets;

b. generating a plurality of optimization inputs drawn at least from a distribution of simulated optimization inputs consistent with the expected return, the standard deviation of return of each of the plurality of assets and the chosen forecast certainty level;

c. computing a simulated mean-variance efficient frontier, the frontier having at least one simulated mean-variance efficient portfolio, for each of the plurality of optimization inputs;

d. associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios for creating a set of associated mean-variance efficient portfolios;

e. establishing a statistical mean for each set of associated mean-variance efficient portfolios, thereby generating a plurality of statistical means, the plurality of statistical means defining a meta-resampled efficient frontier; and

f. selecting a portfolio weight for each asset from the meta-resampled efficient frontier according to a specified investment objective for defining the optimal portfolio subject to the specified investment objective;

wherein acts b. through f. are performed by means of a computer system.

2. A method according to claim **1**, wherein the specified investment objective is a risk objective.

3. A method according to claim **1**, wherein the step of choosing a forecast certainty level from a collection of forecast certainty levels includes choosing from a set of indices from 1 to N, where each index is calibrated to provide a different level of forecast certainty.

4. A method according to claim **3**, further including increasing the number of samples drawn from a distribution based on a higher level of forecast certainty.

5. A method according to claim **3**, wherein each level represent a geometric increase (or decrease) in the number of observations drawn from the distribution.

6. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes associating on the basis of proximity to a maximized expected utility.

7. A method according to claim **6**, wherein the expected utility is a quantity $\mu-\lambda\sigma^2$.

8. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes associating on the basis of a measure of risk.

9. A method according to claim **8**, wherein the measure of risk is variance.

10. A method according to claim **8**, wherein the measure of risk is residual risk.

11. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes associating on the basis of a measure of return.

12. A method according to claim **11**, wherein the measure of return is expected return.

13. A method according to claim **11**, wherein the measure of return is expected residual return.

14. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes associating on a basis of proximity of the portfolios to other simulated mean-variance efficient portfolios.

15. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes associating on a basis of ranking by a specified criterion, the criterion chosen from a group including risk and expected return.

16. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes associating on a basis of investment relevance of the simulated mean-variance efficient portfolios.

17. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes associating simulated portfolios based on a selection of relevant simulations.

18. A method according to claim **1**, wherein the step of associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios includes specifying a fraction of portfolios per simulation to be considered in the association.

19. A method according to claim **1**, further including calculating an estimated normal range of asset weights for an optimal portfolio.

20. A method according to claim **1**, further comprising calculating a portfolio rebalancing probability for a given portfolio with respect to an optimal portfolio.

21. A method according to claim **1**, further comprising investing funds in accordance with the selected portfolio weights.

22. A computer program product for use on a computer system for selecting a value of portfolio weight for each of a specified plurality of assets of an optimal portfolio, the value of portfolio weight chosen from values between two specified real numbers, each asset having an expected return and a standard deviation of return, and each asset having a covariance with respect to each of every other asset of the plurality of assets, the computer program product comprising a computer usable medium having computer readable program code thereon, the computer readable program code including:

a. a routine for generating a number of optimization inputs drawn from a distribution of simulated optimization inputs statistically consistent with the expected return and the standard deviation of return of each of the plurality of assets wherein the number of optimization inputs is based on one of a specified level of estimation certainty and a specified level of forecast certainty;

b. program code for computing a simulated mean-variance efficient frontier, the frontier comprising at least one simulated mean-variance efficient portfolio for each of the plurality of optimization inputs;

c. program code for associating each simulated mean-variance efficient portfolio with a specified set of portfolios for creating a set of associated mean-variance efficient portfolios;

d. a module for establishing a statistical mean for each set of associated mean-variance efficient portfolios, the plurality of statistical means defining a meta-resampled efficient frontier; and

e. program code for selecting a portfolio weight for each asset from the meta-resampled efficient frontier according to a specified risk objective.

23. A computer program product according to claim **22**, wherein the program code for associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios associates on a basis of proximity to a maximized expected utility.

24. A computer program product according to claim **22**, wherein the program code for associating each mean-variance efficient portfolio with a specified set of simulated mean-variance efficient portfolios associates on a basis of ranking by a specified criterion, the criterion chosen from a group including risk and expected return.

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